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LETTER TO THE EDITOR

The asymmetric exclusion model with sequential update

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Abstract. We present a solution for the stationary state of an asymmetric exclusion model with sequential update and open boundary conditions. We solve the model exactly for random hopping in both directions by applying a matrix-product formalism which was recently used to solve the model with sublattice-parallel update (Hinrichsen H 1995 *Weizmann Inst. Preprint* cond-mat/9512172). It is shown that the matrix-algebra describing the sequential update and sublattice-parallel update are identical and can be mapped onto the random sequential case treated by Derrida *et al* (Derrida B, Evans M R, Hakim V and Pasquier V 1993 *J. Phys. A: Math. Gen.* **26** 1493).

The one-dimensional asymmetric exclusion model (AEM) is one simple example of a reaction–diffusion model and has been used to describe various problems in different fields of interest, like the kinetics of biopolymerization [1] and traffic [2]. Using recursion relations on the system size it was solved in 1992 [3] for the case of random sequential update and open boundary conditions. Open boundaries here and in the following mean that particles are injected at one end of a chain of L sites with probability α and removed at the other end with probability β. This model was then solved again by Derrida *et al* 1993 [4] using a matrix product ansatz (MPA), inspired by the matrix product ground state for quantum spin chains with ground-state energy zero [5], for the weights of the stationary configurations. This ansatz was most elegantly used to obtain expressions for the density profile and higher correlations consisting only of products of two matrices (belonging to the two possible states of each site) and two vectors describing the influence of the boundaries. Since then the MPA was extended to also find the transient of the model [6] and to recover solutions of certain known integrable reaction–diffusion models [7]. All these models work with random sequential update.

Hinrichsen [8] solved the AEM model with sublattice-parallel update and was able to confirm earlier conjectures for the correlation functions [9]. It should be made clear that this update is substantially different from the fully parallel update used, for example, for modelling traffic flow. Nevertheless, to our knowledge this is the first model with non-continous time which has been solved using the MPA. The classes of models on which the MPA can be used successfully remains an open question.

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In this letter we study the AEM with *sequential* update. The model then can be defined as follows.

Consider sites located on a chain of length L. Each site i ($0 \leqslant i \leqslant L$) can be occupied by a particle ($\tau_i = 1$) or may be empty ($\tau_i = 0$). We start the update at the right end of the chain and remove a particle at site $i = L$ with probability β. We then look at the pair of sites ($i = L - 1, i = L$). If we find a particle at site $L - 1$ and no particle (called a hole) at site L we move the particle one site to the right with probability p. In the opposite case the particle hops one site to the left with probability q. In the remaining two cases nothing happens. We continue the update with the pair ($i = L-2, i = L-1$) and so forth until we reach the left end of the chain. After the update of pair ($i = 1, i = 2$) we inject a particle at site $i = 1$ with probability α if the site is empty.

Let us remark at this point that this model put on a ring (no injection/removing of particles and periodic boundary conditions) has a trivial stationary state [10, 11] where correlations are absent.

One could reverse the order of the sequential update and go from the left to the right through the chain (using the same rule as before for the pair-update). These two models are connected by a particle–hole symmetry: injecting particles can be seen as removing holes, and vice versa. Therefore it is sufficient to study just one model. The particle–hole symmetry gives a first hint at the phase diagram: one would naively expect the same phase diagram for both models. This implies a symmetry of the phase diagram in α and β.

Following the MPA, we write the stationary probability distribution $P_0(\tau_1, \tau_2, \ldots, \tau_L)$ as

$$P_0(\tau_1, \tau_2, \ldots, \tau_L) = Z_L^{-1} \langle W| \prod_{i=1}^{L} (\tau_i D + (1 - \tau_i) E) |V\rangle \tag{1}$$

or formally as

$$|P_0\rangle = Z_L{}^{-1} \, \langle W| \begin{pmatrix} E \\ D \end{pmatrix}^{\otimes L} |V\rangle \,. \tag{2}$$

The square matrices E and D can be infinite-dimensional. The vectors $\langle W|$ and $|V\rangle$ act in the same vector space as E and D and will, like E and D, depend on α and β. The normalization constant is given by $Z_L = \langle W|(D + E)^L|V\rangle$. $|P_0\rangle$ represents the weights of all the configurations in the stationary state. This implies that $|P_0\rangle$ is invariant under the action of the update-operator or transfer matrix T:

$$T \, |P_0\rangle = |P_0\rangle \tag{3}$$

Let us now write down T explicitly. The boundary conditions may be represented by operators $\mathcal{R}$ and $\mathcal{L}$ acting on site $i = L$ and $i = 1$, respectively:

$$\mathcal{R} = \begin{pmatrix} 1 & \beta \\ 0 & 1 - \beta \end{pmatrix} \qquad \mathcal{L} = \begin{pmatrix} 1 - \alpha & 0 \\ \alpha & 1 \end{pmatrix}. \tag{4}$$

The chosen basis for $\mathcal{R}$ and $\mathcal{L}$ is $(0, 1)$. The update-rule for any pair of sites $(i, i + 1)$ can be written as

$$\mathcal{T}_i = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 - q & p & 0 \\ 0 & q & 1 - p & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{5}$$

The basis is (00, 01, 10, 11) and formally we have

$$T = L \cdot T_1 \cdot \ \cdots \ \cdot T_{(L-1)} \cdot R \tag{6}$$

with

$$L = \mathcal{L} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} \tag{7}$$

$$R = \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes \mathcal{R} \tag{8}$$

$$T_i = \mathbf{1} \otimes \mathbf{1} \cdots \otimes \mathcal{T}_i \otimes \mathbf{1} \cdots \otimes \mathbf{1} \tag{9}$$

where **1** denotes the identity matrix.

It is peculiar that one can use precisely the same mechanism which was used to determine the stationary state of the model with sublattice-parallel update [8] in order to fulfil equation (3):

$$\begin{aligned} &\mathcal{T}\left[\begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} \hat{E} \\ \hat{D} \end{pmatrix}\right] = \begin{pmatrix} \hat{E} \\ \hat{D} \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} \\ &\langle W|\mathcal{L}\begin{pmatrix} \hat{E} \\ \hat{D} \end{pmatrix} = \langle W|\begin{pmatrix} E \\ D \end{pmatrix} \qquad \mathcal{R}\begin{pmatrix} E \\ D \end{pmatrix}|V\rangle = \begin{pmatrix} \hat{E} \\ \hat{D} \end{pmatrix}|V\rangle \end{aligned} \tag{10}$$

with some square matrices $\hat{E}, \hat{D}$. This means that a 'defect' is created in the beginning of an update at site $i = L$ and then transported through the chain until it reaches the left end, where it disappears.

Equation (10) leads to the following bulk algebra:

$$\begin{aligned} &[E, \hat{E}] = [D, \hat{D}] = 0 \\ &(1-q)E\hat{D} + pD\hat{E} = \hat{E}D \\ &qE\hat{D} + (1-p)D\hat{E} = \hat{D}E \end{aligned} \tag{11}$$

and the boundary conditions

$$\begin{aligned} &\langle W|\hat{E}(1-\alpha) = \langle W|E \qquad (1-\beta)D|V\rangle = \hat{D}|V\rangle \\ &\langle W|(\alpha\hat{E} + \hat{D}) = \langle W|D \qquad (E + \beta D)|V\rangle = \hat{E}|V\rangle\,. \end{aligned} \tag{12}$$

For $q = 0$, $p = 1$ we recover the algebra which was solved in [8] for the model with sublattice parallel-update, and we can adopt the two-dimensional representations of $E, D, \hat{E}, \hat{D}, \langle W|, |V\rangle$ of the algebra (11), even though $|P_0\rangle$ has a different structure. One can easily show that the density profile of the sequential update corresponds to the density of the even sites with sublattice-parallel update and one gets essentially the same phase diagram.

One can check that for the case

$$(1-\alpha)(1-\beta)(1-q) = 1 - p \tag{13}$$

there exists a one-dimensional solution of the complete algebra. This equation defines the lines in the phase diagram on which the mean field solution becomes exact. One can use these lines to calculate different currents in the phase diagram [10, 11] and we can exclude the case $p = q = 1$ in what follows.

To solve the general algebra, we first note that

$$[E + D, \hat{E} + \hat{D}] = 0 \tag{14}$$

holds for all values of p, q. By demanding

$$\hat{E} = E + \lambda\mathbf{1} \tag{15}$$

$$\hat{D} = D - \lambda\mathbf{1} \tag{16}$$

(with some real number λ and the identity matrix $\mathbf{1}$) one can reduce the whole algebra of seven equations to just three equations:

$$\begin{aligned} & pDE - qED = \lambda(1-q)E + \lambda(1-p)D \\ & \alpha\langle W|E = \lambda(1-\alpha)\langle W| \\ & \beta D|V\rangle = \lambda|V\rangle. \end{aligned} \tag{17}$$

We define

$$\begin{aligned} & \widetilde{D} := \lambda(1-p)D \\ & \widetilde{E} := \lambda(1-q)E \\ & \lambda^2 := \frac{1}{(1-q)(1-p)} \end{aligned} \tag{18}$$

and rewrite (17) as

$$\begin{aligned} & p\widetilde{D}\widetilde{E} - q\widetilde{E}\widetilde{D} = \widetilde{E} + \widetilde{D} \\ & \alpha(1-p)\langle W|\widetilde{E} = (1-\alpha)\langle W| \\ & \beta(1-q)\widetilde{D}|V\rangle = |V\rangle. \end{aligned} \tag{19}$$

This is the algebra for the AEM with random sequential update but the same local transfer matrix and the same boundary conditions as in our model. It was solved by Derrida *et al* [4] with infinite-dimensional matrices. Note that one has to rescale the vectors $\langle W|$ and $|V\rangle$ of their solution with $(1-\alpha)/(1-p)$ and $1/(1-q)$, respectively.

For the case $q = 0$ we write down a slightly different representation ($\lambda = 1$):

$$D = \frac{1}{p}\begin{pmatrix} \frac{p}{\beta} & a_1 & 0 & 0 & \cdot \\ 0 & 1 & 1 & 0 & \cdot \\ 0 & 0 & 1 & 1 & \cdot \\ 0 & 0 & 0 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \qquad E = \frac{1}{p}\begin{pmatrix} \frac{p(1-\alpha)}{\alpha} & 0 & 0 & 0 & \cdot \\ a_2 & 1-p & 0 & 0 & \cdot \\ 0 & 1-p & 1-p & 0 & \cdot \\ 0 & 0 & 1-p & 1-p & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \tag{20}$$

$$\langle W| = (1, 0, 0, 0) \qquad |V\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{21}$$

$$a_1 a_2 = \frac{p}{\alpha\beta}[(1-p) - (1-\alpha)(1-\beta)]\,. \tag{22}$$

One sees that the constraint (13) leads to an effectively one-dimensional representation as expected.

The MPA now makes it straightforward to calculate the density profile, higher correlations, the current and the phase diagram. We note at this point that for the case $q = 0$ the randomness in p produces qualitatively the same phase diagram as in the case of random sequential dynamics with $q = 0$, $p = 1$. This is understandable for small p but far from obvious for general p [10, 11]. Further results, also concerning the comparison of the AEM with random sequential, sublattice-parallel, sequential and fully parallel update will be published elsewhere [10, 11].

The present letter shows that the sequential AEM with random hopping in both directions can be solved by means of a mapping onto the known solution of the random sequential AEM. This also implies a generalization of the known solution of the AEM for sublattice-parallel update, which had been solved for deterministic hopping. It is remarkable that $pDE - qED = E + D$ is the fundamental bulk algebra equation for all three types of update discussed so far. This opens the possibility to understand more about the nature and implications of different types of update and randomness [10, 11].

It would be most interesting to find out if a more general connection exists between time-continous and time-non-continous models. The solution of the sequential dynamics gives reason to hope that the MPA is capable of solving the fully parallel case, which can be written as a three-state model with the same sequential update as studied in this work [10, 11].

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